# A Real-Imaginative Guide to Complex Numbers

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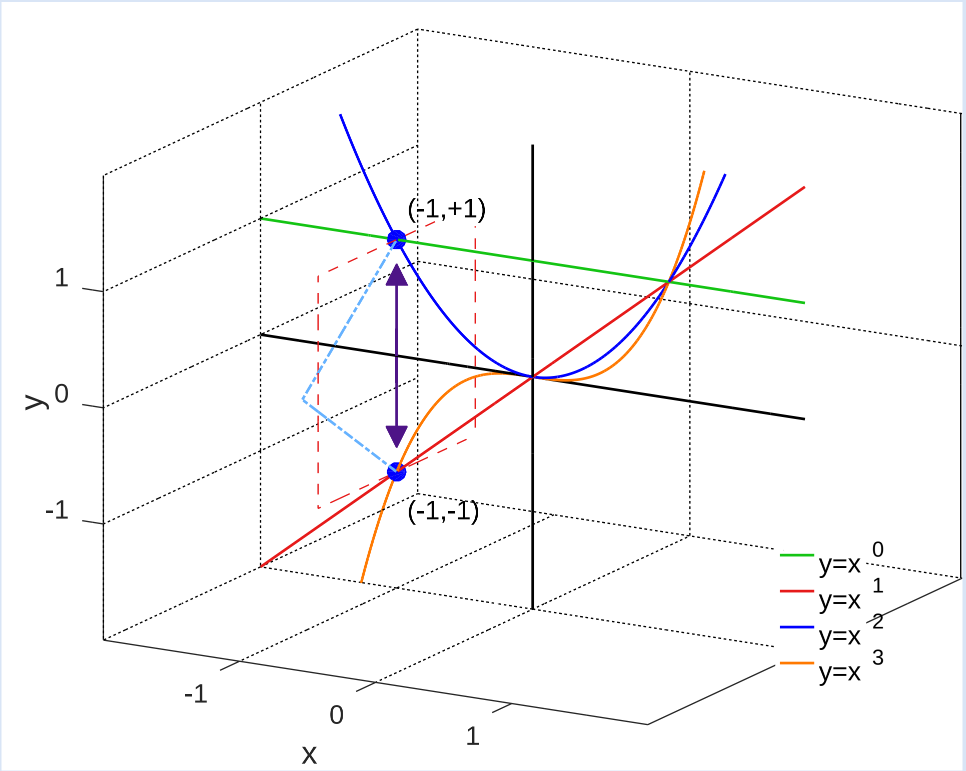


**Wireless Pi**

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On a cold morning in August 2015, I narrowly missed a train to my office in Melbourne city. With nothing else to do in the next 20 minutes, my mind wandered towards an intuitive view of complex numbers, something that has puzzled me since long. In particular, I wanted to seek answers to the following questions.

**(a)** What is the role of the number $\sqrt{-1}$ in mathematics? What sets it apart from other impossible numbers, e.g., a number $k$ such that $|k| = -1$?

**(b)** Why is $\sqrt{-1}$ a rotation by 90° in a 2D plane? A square root and a rotation do not seem to be related at all.

**(c)** Why is the expression $e^{i\theta}$ a rotation of 1 by $\theta$ radians on a unit circle? Is it possible to make sense out of a number like $2.71828^{\sqrt{-1}\cdot\theta}$?

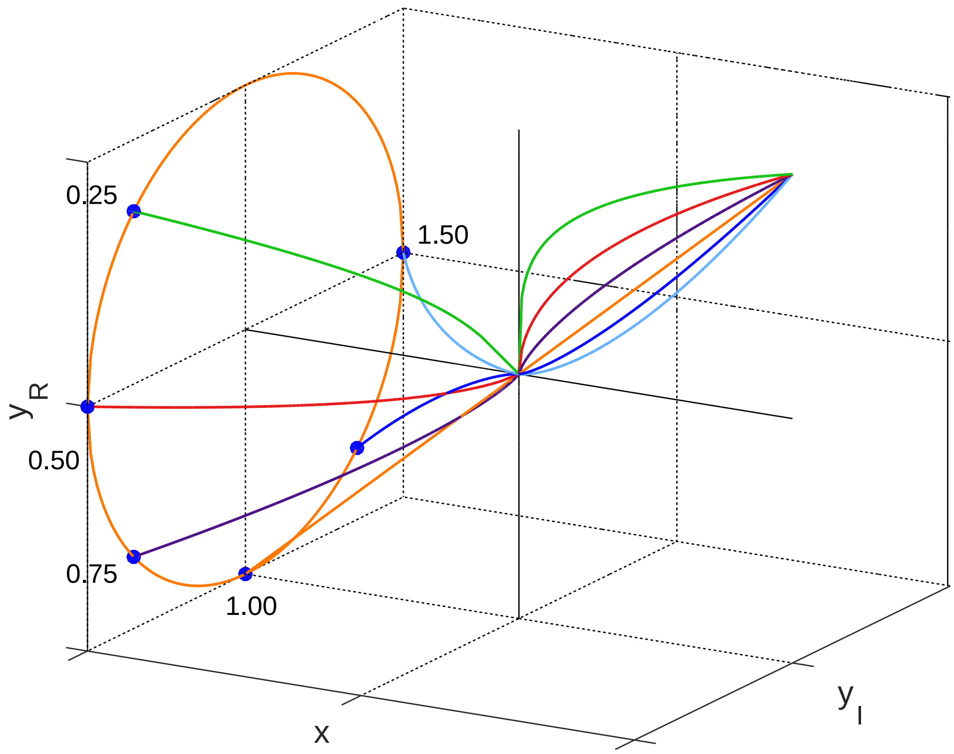
The ideas I wrote were forgotten in my notebook but this puzzle was reignited in my mind a few weeks ago. By the time you finish this guide, you will have answers to all three questions above and your way of looking at complex numbers will probably change forever. It should be kept in mind, however, that complex numbers can still be utilized without using any of these constants. In my book *Wireless Communications from the Ground Up – An SDR Perspective*, for example, I have not used $e$, $i$ or $j$ and still covered in depth how Digital Signal Processing (DSP) is applied to the design of wireless communication systems – a field known as Software Defined Radio (SDR).

## 1 Words from the Past

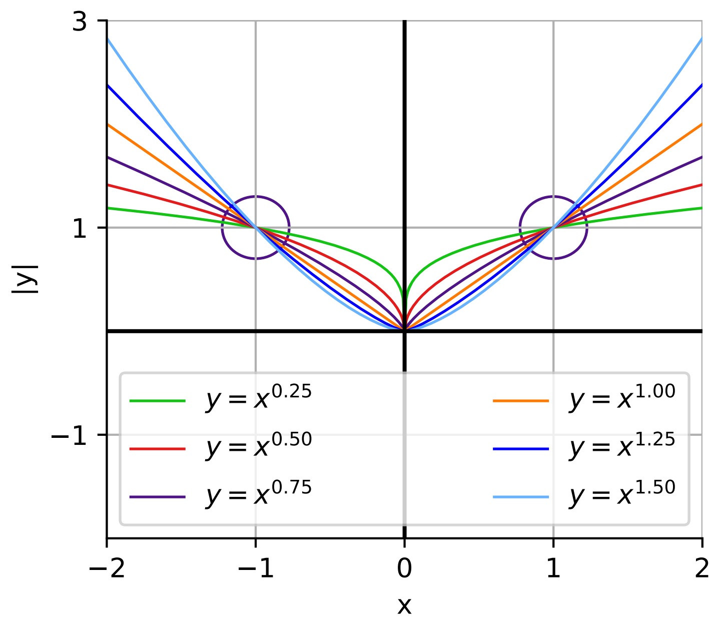
The history of imaginary numbers can be summarized in three quotes from three greatest mathematicians of their respective centuries.

Figure 1: Descartes, Euler and Gauss

- The square roots of negative numbers were called sophisticated or subtle before the publication of La Geometrie by the French mathematician René Descartes in 1637. He named them imaginary due to the impossibility of having a geometric construction for them and ignored possibly complex solutions to his equations. However, at the very last line of La Geometrie, he said: "I hope that posterity will judge me kindly, not only as to things which I have explained, but also as to those which I have intentionally omitted so as to leave to others the pleasure of discovery".
- Then, we have Leonhard Euler writing in his Algebra of 1770: "All such expressions as $\sqrt{-1}$, $\sqrt{-2}$, etc., are consequently impossible or imaginary numbers, since they represent roots of negative quantities; and of such numbers we may truly assert that they are neither nothing, nor greater than nothing, nor less than nothing, which necessarily constitutes them imaginary or impossible."
- Finally, Carl Friedrich Gauss wrote in 1831: "If this subject has hitherto been considered from the wrong viewpoint and thus enveloped in mystery and surrounded by darkness, it is largely an unsuitable terminology which should be blamed. Had $+1$, $-1$ and $\sqrt{-1}$ instead of being called positive, negative




(a) Fractional powers arranged around the unit circle in the negative half




(b) Magnitude plots $|y|$ for curves on the left with circles on $(-1, +1)$ and $(+1, +1)$

Figure 19: Fractional powers of $x = 0.25$, 0.5, 0.75, 1, 1.25 and 1.75. Their magnitudes exhibit an even symmetry beautifully crafted through imaginary numbers

This chain continues for an infinitely many beanstalks in the magical garden. The giant will only die if he falls from a height of 2.72 imaginary meters. Which beanstalk should Jill climb?

Being good at mathematics, Jill computes the resulting sequence as follows.

$$\begin{aligned}\left(1+\frac{1}{1}\right)^{1} &= 2.00\\ \left(1+\frac{1}{2}\right)^{2} &= 2.25\\ \left(1+\frac{1}{3}\right)^{3} &= 2.37\\ &\vdots\\ \left(1+\frac{1}{1000000}\right)^{1000000} &= 2.718280469\end{aligned}$$

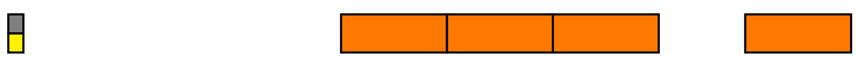
She concludes that no beanstalk, even in a magical garden and with continuous growth, will reach a height of 2.72 imaginary meters and walks away from the riches. This limiting value is the constant $e$.

$$e^1 = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2.718281828459045$$

It is subsequently deduced that $e$ represents a continuous growth for a unit period of time at a rate of 100%. What is overlooked here is the following property.

The real power of $e$ becomes visible by writing the above equation in the form

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)\cdots\cdots\left(1+\frac{1}{n}\right)$$



Like a machine press, the constant $e$ compresses an infinite number of multiplications into a single quantity!

Figure 21: Like a machine press, the constant $e$ has the power to compress an infinite number of multiplications into a single quantity

# About the Author

I write about wireless communications and software defined radio with a focus on beautiful figures, simple mathematics and intuitive reasoning. I am the author of the book Wireless Communications from the Ground Up – An SDR Perspective on software defined radios which is available at my website https://wirelesspi.com along with a video course with practical exercises to implement the concepts.

After obtaining a PhD in Electrical Engineering from Texas A&M University, I have mostly worked in different research labs on DSP algorithms development for real world demonstrations of wireless systems such as a MIMO-OFDM testbed, a low-SNR receiver and phase of arrival based localization. As a dad of 3 kids, my time is spent on slowly emptying the queue of projects that entertain me the most. You can contact me at info@wirelesspi.com.